3.1: Symmetry and Coordinate Graphs

**Essential Questions:**
- How do we determine symmetry using algebra?
- How do we classify functions as even or odd?

**Point Symmetry**
- The origin is a common point of symmetry.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>f(x) = x^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

(Note that f(-x) = f(x).)

**Line Symmetry**
- Each graph below has line symmetry. The equation of each line of symmetry is given. Graphs that have line symmetry can be folded along the line of symmetry so that the two halves match exactly.

**Symmetry**
- **x-axis**
  What we do: keep x the same but negate y.

\[ x = y^2 - 3 \]

\[ x = (-y)^2 - 3 \]

Equates original function.
**Symmetry**

**y-axis**

What we do: keep y the same but negate x

\[ y = -x^2 + 3 \]

\[ y = -(x)^2 + 3 \]

\[ y = -x^2 + 3 \]

**y = -x**

What we do: interchange AND negate x and y

\[ y = \frac{6}{x} \]

\[ -x = \frac{6}{y} \]

\[ xy = 6 \]

\[ y = \frac{6}{x} \]

**origin**

What we do: negate x AND negate y

\[ y = x^3 - 4x \]

\[ -y = (-x)^3 - 4(-x) \]

\[ -y = -x^3 + 4x \]

\[ y = -x^3 + 4x \]

**Line Symmetry**

<table>
<thead>
<tr>
<th>Symmetry with Respect to Line</th>
<th>Definition and Test</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>x-axis</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rule: (x, y) ↔ (x, -y)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rule: (x, y) ↔ (-x, y)</td>
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<td></td>
</tr>
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</table>

**Rule:**

- If \( (a, b) \) is on the graph, then \( (-a, b) \) is also on the graph. Substituting \( (a, b) \) and \( (-a, b) \) into the equation produces equivalent equations.

**y-axis**

Rule: (x, y) ↔ (-x, y)

**Line Symmetry**

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<tr>
<td><strong>y = x</strong></td>
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<td></td>
</tr>
<tr>
<td>Rule:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x, y) ↔ (y, x)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Rule:**

- If \( (a, b) \) is on the graph, then \( (b, a) \) is also on the graph. Substituting \( (a, b) \) and \( (b, a) \) into the equation produces equivalent equations.
Common types of symmetry are:
- with respect to the \( x \)-axis
- with respect to the \( y \)-axis
- with respect to the origin
- with respect to the line \( y = x \)
- with respect to the line \( y = -x \)

### Table of Symmetric Relationships

<table>
<thead>
<tr>
<th>Symmetry with respect to:</th>
<th>What do we do?</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )-axis</td>
<td>keep ( x ) the same but negate ( y )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( y )-axis</td>
<td>keep ( y ) the same but negate ( x )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( y = x )</td>
<td>Interchange ( x ) and ( y )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( y = -x )</td>
<td>Interchange AND negate ( x ) and ( y )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>origin</td>
<td>Negate ( x ) and negate ( y )</td>
<td>( \uparrow )</td>
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</tbody>
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### Example 1

Determine the types of symmetry for the graph of \( xy = -2 \)

- \( x \)-axis: No
- \( y \)-axis: Yes
- origin: Yes
- \( y = x \): Yes
- \( y = -x \): Yes

### Example 2

Determine the types of symmetry for the graph of \( y^2 = \frac{4x^2}{9} - 4 \)

- \( x \)-axis: Yes
- \( y \)-axis: Yes
- origin: Yes
- \( y = x \): No
- \( y = -x \): No

### Even and Odd Functions

A function \( f \) is **even** if, for every number \( x \) in its domain, the number \(-x\) is also in the domain and

\[
 f(-x) = f(x) 
\]

For an **even** function, for every point \((x, y)\) on the graph, the point \((-x, y)\) is also on the graph.
### Even and Odd Functions

**A function \( f \) is odd if,** for every number \( x \) in its domain, the number \(-x\) is also in the domain and

\[
f(-x) = -f(x)
\]

For an odd function, for every point \((x, y)\) on the graph, the point \((-x, -y)\) is also on the graph.

### Even and Odd Functions

<table>
<thead>
<tr>
<th>Even functions</th>
<th>Odd functions</th>
</tr>
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<tr>
<td>( f(-x) = f(x) )</td>
<td>( f(-x) = -f(x) )</td>
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### Classifying Functions as Even or Odd

- **Even** functions are symmetric with respect to the **y-axis**
- **Odd** functions are symmetric with respect to the **origin**

### Example

Determine whether each graph given is an **even** function, an **odd** function, or a function that is neither even nor odd.

- Even
- Neither
- Odd

### Example

**Identifying Even and Odd Functions**

Use a graphing utility to conjecture whether each of the following functions is even, odd, or neither. Verify the conjecture algebraically. Then state whether the graph is symmetric with respect to the \( y \)-axis or with respect to the origin.

1. \( f(x) = -3x^4 - x^2 + 2 \) **even; y-axis**
2. \( g(x) = 5x^3 - 1 \) **neither**
3. \( h(x) = 2x^3 - x \) **odd; origin**
DO: evens 14-26, 32-36, p. 134

Exercises

Determine whether each graph is symmetric with respect to the origin.
3. Answer: no 4. Answer: yes

5. Complete the graph so that it is the graph of an odd function.

Determine whether the graph of each function is symmetric with respect to the origin.
6. $f(x) = x^2 - 3x$
7. $f(x) = \frac{1}{x} - x^2$

Determine whether the graph of each equation is symmetric with respect to the x-axis, y-axis, the line $y = x$, or none of these.
8. $5x^2 - y = -1$
9. $x^2 + y^2 = 4$

Determine whether the graph of each equation is symmetric with respect to the x-axis, the y-axis, both, or neither. Use the information about symmetry to graph the relation.
11. $y = \sqrt{2 - x^2}$
12. $|y| = x^2$

Determine whether the graph of each equation is symmetric with respect to the x-axis, y-axis, the line $y = x$, or none of these.
14. $f(x) = 3x$
15. $f(x) = x^2 - 1$
16. $f(x) = 5x^2 + 3x + 9$
17. $f(x) = -\frac{1}{x}$
18. $f(x) = -7x^2 + 8x$
19. $f(x) = \frac{1}{x^2} - x^{200}$

20. Is the graph of $g(x) = \frac{x^2 - 1}{x}$ symmetric with respect to the origin? Explain how you determined your answer.

Determine whether the graph of each equation is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these.
21. $x^2 - y^2 = 2$
22. $x + y^2 = 1$
23. $y = 8x$
24. $y = \frac{1}{x}$
25. $x^2 + y^2 = 4$
26. $y^2 = \frac{5x^2}{9} - 4$
27. Which line(s) are lines of symmetry for the graph of $x^2 = \frac{1}{y^2}$?

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Exercises

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